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MADRAS

ON A TEST FOR THE COMPLETENESS (L^2) OF A NORMAL ORTHOGONAL SET AND ITS APPLICATIONS

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1. Steinhaus and Kaczmarz have given the following test on page 95 of their *Theorie der orthogonal Reihen*:—

A necessary and sufficient condition for the completeness (L^2) *of a normal orthogonal* (*O.N.*) *set* $\{\phi_n\}$ *in* (a, b) *is that for* $a \leqslant \alpha \leqslant \beta \leqslant b$,

$$\beta - \alpha = \sum \left\{ \int_\alpha^\beta \phi_n dt \right\}^2 .$$

It is the object of this paper, to derive a simplified form of this test from two general theorems (L_1 and L_2) which are true for any normal orthogonal set not necessarily complete, and by applying this test, to establish by very elementary methods the completeness (L^2) of the Fourier set and of the polynomials of Hermite, Laguerre, and Legendre. The simplified form of the test is as follows:—

A necessary and sufficient condition that an O.N. set defined in $0 \leqslant x \leqslant 1$ *be* (L^2) *complete is that for all* $0 \leqslant a \leqslant 1$

$$a = \sum^{\infty} \left\{ \int_0^a \phi_n dt \right\}^2 . \qquad (C)$$

2. THEOREM L_1.

If $\{\phi_n\}$ *be O.N. in* $(0, 1)$ *and* f *is* L^2 *in* $(0, 1)$ *while* f_n *is such that* $\int_0^1 (f - f_n)^2 dt \to 0$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \sum_{p=1}^{\infty} a_{np}^2 = \sum_{p=1}^{\infty} a_p^2,$$

where

$$a_{np} = (f_n \phi_p) = \int_0^1 f_n \phi_p \, dt$$

and
$$a_p = (f\phi_p) = \int_0^1 f\phi_p dt.$$

PROOF:
$$\left\{ \sum_p a_{np}^2 - \sum_p a_p^2 \right\}^2$$
$$= \left\{ \Sigma(a_{np}-a_p)(a_{np}+a_p) \right\}^2 \leqslant \Sigma(a_{np}-a_p)^2 . \Sigma(a_{np}+a_p)^2. \quad (2.1)$$

By Bessel's inequality
$$\Sigma(a_{np}-a_n)^2 \leqslant \int (f-f_n)^2 dt, \quad (2.2)$$
$$\Sigma(a_{np}+a_n)^2 \leqslant \int (f+f_n)^2 dt, \quad (2.3)$$

and
$$\left\{ \int (f+f_n)^2 dt \right\}^{\frac{1}{2}} = \left\{ \int (2f+f_n-f)^2 dt \right\}^{\frac{1}{2}}$$
$$\leqslant \left\{ \int (2f)^2 dt \right\}^{\frac{1}{2}} + \left\{ \int (f_n-f)^2 dt \right\}^{\frac{1}{2}}. \quad (2.4)$$

Hence
$$\left\{ \sum_p a_{np}^2 - \sum_p a_p^2 \right\}^2$$
$$\leqslant \int (f-f_n)^2 dt . \left\{ \left\{ \int (2f)^2 dt \right\}^{\frac{1}{2}} + \left\{ \int (f_n-f)^2 dt \right\}^{\frac{1}{2}} \right\}^2. \quad (2.5)$$

Since $\int_0^1 (f-f_n)^2 dt \to 0$ as $n \to \infty$, Theorem L_1 is established.

3. THEOREM L_2.

If $\{\phi_n\}$ be any O.N. set in $(0, 1)$, and f_1 and f_2 are such that Parseval's theorem is true for them with regard to $\{\phi_n\}$, then Parseval's theorem is also true for $f_1 \pm f_2$ with regard to $\{\phi_n\}$.

PROOF: Let $s_n = \sum_{r=1}^{n} (f_1\phi_r).\phi_r$. Then by hypothesis,
$$\int_0^1 (f_1-s_n)^2 dt \to 0 \text{ as } n \to \infty \quad (3.1)$$

and
$$\left\{ \int_0^1 f_2(f_1-s_n) dt \right\}^2 \leqslant \int_0^1 f_2^2 dt . \int_0^1 (f_1-s_n)^2 dt \to 0 \text{ as } n \to \infty. \quad (3.2)$$

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$$\text{Hence} \int_0^1 f_2 f_1 dt = \lim_{n\to\infty} \int_0^1 f_2 s_n dt = \sum^{\infty} (f_1\phi_n)\cdot(f_2\phi_n). \tag{3.3}$$

Now $\Sigma(\overline{f_1 \pm f_2 \cdot \phi_n})^2$

$$= \Sigma(f_1\phi_n)^2 + \Sigma(f_2\phi_n)^2 \pm 2\Sigma(f_1\phi_n)\cdot(f_2\phi_n). \tag{3.4}$$

By hypothesis and (3.3), this is equal to

$$\int_0^1 (f_1 \pm f_2)^2 dt. \tag{3.5}$$

By induction, Theorem L_2 can be extended as follows.

THEOREM $L_2(a)$.

If Parseval's theorem is true for $f_1, f_2, \ldots, f_k$, it is also true for the linear combination $\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k$.

4. We shall now deduce (C).

Necessity of condition (C). If the system be complete (L^2), then Parseval's theorem is true, and we consider $f(x)$ defined as follows:—

$$f(x) = 1 \text{ in } 0 \leqslant x \leqslant a,\ f(x) = 0 \text{ in } a < x \leqslant 1.$$

$$\text{Then} \qquad a = \int_0^1 f^2 dx = \sum (f\phi_n)^2 = \sum \left\{ \int_0^a \phi_n dt \right\}^2, \tag{4.1.}$$

thus establishing the necessity of condition (C).

Sufficiency of condition (C).

If $f(x)$ is L^2, it is possible to divide $(0,1)$ into a set of non-overlapping intervals by points $0 < x_1 < x_2 < \ldots < x_{n-1} < 1 = x_n$ such that we have a function $f_\varepsilon(x)$ taking the constant value M_r in $x_{r-1} \leqslant x < x_r$ and $\int_0^1 (f - f_\varepsilon)^2 dx \leqslant \varepsilon$.

Let $F_r(x) = 1$ in $0 \leqslant x < x_r$; $F_r(x) = 0$ if $x_r \leqslant x \leqslant 1$.

$$\text{Then} \qquad f_\varepsilon = \Sigma M_r (F_r - F_{r-1}). \tag{4.2}$$

Since condition (C) is given, Parseval's theorem is true for every f_r.

By Theorem $L_2(a)$, Parseval's theorem is true for $f_\varepsilon(x)$, since

$$f_\varepsilon = \Sigma M_r (F_r - F_{r-1}),$$

i.e.
$$\Sigma(f_\varepsilon\phi_n)^2=\int_0^1 f_\varepsilon^2 dx. \tag{4.3}$$

By hypothesis
$$\int_0^1 f_\varepsilon^2 dx \to \int_0^1 f^2\, dx \text{ as } \varepsilon\to 0$$

since
$$\int_0^1 (f-f_\varepsilon)^2\, dx \leqslant \varepsilon. \tag{4.4.}$$

By Theorem L_1, $\int (f_\varepsilon\phi_n)^2 \to \sum (f\phi_n)^2$ as $\varepsilon\to 0$. (4.5)

Hence from (4.3), (4.4) and (4.5)
$$\Sigma(f\phi_n)^2=\int_0^1 f^2\, dx, \tag{4.6}$$

thus establishing the sufficiency of condition (C).

NOTE 1. It may be noted that Bessel's inequality implies by itself
$$\sum^{\infty}\left\{\int_0^a \phi_n dt\right\}^2 \leqslant a. \tag{4.7}$$

NOTE 2.* The complete equivalence of test (C) and of the test of Steinhaus given in **1** follows from Theorem L_2.

NOTE 3*. If (C) is satisfied for an enumerable everywhere dense set of values of a in $(0, 1)$ the condition (C) still remains necessary and sufficient. This follows from Theorem L_1.

5. *Applications.* The condition (C) is easily proved for the Fourier set $\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}, \frac{\cos nx}{\pi^{\frac{1}{2}}}, \frac{\sin nx}{\pi^{\frac{1}{2}}}\right\}$ in $0\leqslant x\leqslant 2\pi$, and for the normalized Legendre polynomials in $-1\leqslant x\leqslant 1$.

(*a*) *Fourier set.* In this case, the condition (C) to be proved is
$$\frac{x^2}{2\pi}+\sum_1^{\infty}\frac{\sin^2 nx+(1-\cos nx)^2}{n^2\pi}=x,$$

or
$$\frac{x^2}{2\pi}+\frac{2}{\pi}\sum_1^{\infty}\frac{(1-\cos nx)}{n^2}=x. \tag{5.1}$$

*I am thankful to the referee for Notes 2 and 3.

Let $$F_n(x) = \frac{x^2}{2\pi} + \frac{2}{\pi}\sum_{r=1}^{n} \frac{1-\cos rx}{r^2}$$

and $F(x) = \lim F_n(x)$ as $n \to \infty$.

Let $0 < a < b < 2\pi$. Then by easy calculation

$$F_n(b) - F_n(a) = \frac{1}{\pi}\int_a^b dt \int_0^t \frac{\sin (2n+1)u/2}{\sin u/2}\, du$$

$$= \frac{4}{\pi}\int_{\frac{a}{2}}^{\frac{b}{2}} dt \int_0^t \frac{\sin (2n+1)u}{\sin u}\, du. \tag{5.2}$$

Now $$\int_0^t \frac{\sin (2n+1)u}{\sin u}\, du$$

$$= \int_0^t \sin (2n+1)u \left\{ \frac{1}{\sin u} - \frac{1}{u} \right\} + \int_0^t \frac{\sin (2n+1)u}{u}\, du$$

$$= I_1 + I_2.$$

$\left\{ \frac{1}{\sin u} - \frac{1}{u} \right\}$ has a bounded derivative in $0 \leqslant u \leqslant \frac{b}{2} < \pi$.

Hence $I_1 = O\left(\frac{1}{n} \right)$ for all $0 \leqslant t \leqslant \frac{b}{2} < \pi$. (5.4)

$$I_2 = \int_0^{(2n+1)t} \frac{\sin u}{u}\, du = \left\{ \int_0^\infty - \int_{(2n+1)t}^\infty \right\} \frac{\sin u}{u}\, du$$

$$= \frac{\pi}{2} + O\left(\frac{1}{na} \right), \text{ since } t \geqslant \frac{a}{2} > 0. \tag{5.5}$$

Hence

$$F_n(b) - F_n(a) = b - a + O\left(\frac{1}{n} \right),$$

and $F(b) - F(a) = b - a$, or $F(b) - b =$ constant, i.e.

$$F(x) - x = \text{constant for all } 0 < x < 2\pi. \tag{5.6}$$

Now the terms defining the series for $F(x)$ are $O\left(\frac{1}{n^2} \right)$ and are continuous in $0 \leqslant x \leqslant 2\pi$. Hence $F(x) - x$ is continuous in $0 \leqslant x \leqslant 2\pi$.

Also $F(x) - x = F(0) = 0$ for all x in $0 \leqslant x \leqslant 2\pi$.

Thus (5.1) is established.

(*b*) *Legendre Polynomials.*

Taking $\phi_n = \left(\frac{2n+1}{2} \right)^{\frac{1}{2}} P_n$ in $-1 \leqslant x \leqslant 1$, the condition (*C*) in this case (to be proved) will be:—

$$\sum_{0}^{\infty} \frac{2n+1}{2} \left\{ \int_{-1}^{x} P_n \, dt \right\}^2 = x+1. \tag{5.7}$$

Now, $P'_{n+1} - P'_{n-1} = (2n+1)P_n$, $P_0 = 1$, $P_1 = x$, so that

(5.7) can be written as $\dfrac{(1+x)^2}{2} + \frac{1}{2} \sum_{1}^{\infty} \dfrac{(P_{n+1} - P_{n-1})^2}{2n+1} = x+1.$ (5.8)

Simplifying, it becomes $P_1^2 + \sum_{1}^{\infty} \dfrac{(P_{n+1} - P_{n-1})^2}{2n+1} = 1.$ (5.9)

Let $F_n(x) = P_1^2 + \sum_{r=1}^{n} \dfrac{(P_{r+1} - P_{r-1})^2}{2r+1}.$

Then $F'_n(x) = 2\left\{ P_1 P_0 + \sum_{1}^{n} (P_{r+1} - P_{r-1}) P_r \right\} = 2 P_n P_{n+1}.$ (5.10)

Hence $|F_n(x) - 1| = |F_n(x) - F_n(-1)| = \left| \int_{-1}^{x} F'_n(x) \right| dx$

$$= 2 \left| \int_{-1}^{x} P_n P_{n+1} \, dt \right|$$

$$\leqslant 2 \left\{ \int_{-1}^{+1} P_n^2 \, dt \int_{-1}^{+1} P_{n+1}^2 \, dt \right\}^{\frac{1}{2}} = \frac{4}{[(2n+1)(2n+3)]^{\frac{1}{2}}}$$

Hence (5.9) is established.

6. *Hermite Polynomials.*

We shall first prove condition (*C*) for the system $\left\{ K_n H_n \right\}$, where $K_n^2 = \dfrac{1}{2^n \pi^{\frac{1}{2}} n!}$ and $H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2})$, H_n being considered as a function of u where $u = \int_{-\infty}^{x} e^{-t^2} \, dt$, in $0 \leqslant u \leqslant \pi^{\frac{1}{2}}$.

Then we shall deduce Parseval's theorem for the system $\left\{ e^{-x^2/2} K_n H_n \right\}$ in $-\infty \leqslant x \leqslant \infty$. Now

$$\int_{-\infty}^{\infty} H_n H_m \, e^{-x^2} dx = \int_{0}^{\pi^{\frac{1}{2}}} H_n H_m \, du = 0, \; n \neq m$$

and

$$\int_{-\infty}^{\infty} H_n^2 e^{-x^2} dx = \int_{0}^{\pi^{\frac{1}{2}}} H_n^2 \, du = \frac{1}{K_n^2}.$$

Hence the system $\{K_n H_n\}$ considered as functions of u is $O.N.$ in $0 \leqslant u \leqslant \pi^{\frac{1}{2}}$. (6.1.)

The condition (C) in this case to be proved will be

$$\sum\left\{\int_0^u K_n H_n dt\right\}^2 = u. \tag{6.2}$$

Since $H_0 = 1$ and $\int_{-\infty}^{x} H_n e^{-t^2} dt = -e^{-x^2} H_{n-1}$,

(6.2) becomes $\pi^{-\frac{1}{2}}u^2 + e^{-2x^2}\sum_1^{\infty} K_n^2 H_{n-1}^2 = u.$ (6.3)

Let $$F_n(u) = \pi^{-\frac{1}{2}}u^2 + e^{-2x^2}\sum_{r=1}^{n} K_r^2 H_{r-1}^2.$$

We shall now establish

$$F_n(u) - u + \frac{\pi^{\frac{1}{2}}}{2} - F_n\left(\frac{\pi^{\frac{1}{2}}}{2}\right) = 2\int_0^x e^{-t^2} dt \int_0^t K_n^2 H_n^2 e^{-v^2} dv = I_n. \tag{6.4}$$

Proof of (6.4).

$$\frac{dF_n}{du} = 2\pi^{-\frac{1}{2}}u + e^{x^2}\frac{d}{dx}\left(e^{-2x^2}\sum_1^n K_r^2 H_{r-1}^2(x)\right)$$

$$= 2\pi^{-\frac{1}{2}}u + 2e^{-x^2}\sum_{r=1}^{n} K_r^2\left\{H_{r-1}\left(\frac{d}{dx}H_{r-1}\right) - 2xH_{r-1}^2\right\}$$

$$= 2\pi^{-\frac{1}{2}}u - 2e^{-x^2}\sum_{r=1}^{n} K_r^2 H_{r-1} H_r, \tag{6.5}$$

since $$\frac{dH_n}{dx} - 2xH_n = -H_{n+1}$$

$$\frac{d^2 F_n}{du^2} = 2\pi^{-\frac{1}{2}} - e^{x^2}\frac{d}{dx}\left\{2e^{-x^2}\sum_{r=1}^{n} K_r^2 H_{r-1} H_r\right\}$$

$$= 2\pi^{-\frac{1}{2}} - 2\sum_{r=1}^{n} K_r^2\left\{H_{r-1}\frac{dH_r}{dx} + H_r\frac{dH_{r-1}}{dx} - 2xH_r H_{r-1}\right\}. \tag{6.6}$$

Since $\frac{dH_n}{dx}=2nH_{n-1}$ and $\frac{dH_n}{dx}-2xH_n=-H_{n-1}$, (6.6) becomes

$$=2\pi^{-\frac{1}{2}}-2\sum_{r=1}^{n}K_r^2(2rH_{r-1}^2-H_r^2)$$

$$=2\pi^{-\frac{1}{2}}-2\sum_{r=1}^{n}(K_{r-1}^2H_{r-1}^2-K_r^2H_r^2)$$

$$=2K_n^2H_n^2. \tag{6.7}$$

Since $H_n(0)=0$ for all odd values of n, $\frac{dF_n}{du}\left(\frac{\pi^{\frac{1}{2}}}{2}\right)=1$, and we get

$$F_n(u)-u+\frac{\pi^{\frac{1}{2}}}{2}-F_n\left(\frac{\pi^{\frac{1}{2}}}{2}\right)$$

$$=F_n(u)-F_n\left(\frac{\pi^{\frac{1}{2}}}{2}\right)-\left(u-\frac{\pi^{\frac{1}{2}}}{2}\right)\frac{dF_n}{du}\left(\frac{\pi^{\frac{1}{2}}}{2}\right)$$

$$=\int_{\pi^{\frac{1}{2}}/2}^{u_1}dv\int_{\pi^{\frac{1}{2}}/2}^{v}\frac{d^2F_n}{du^2}du$$

$$=2\int_{\pi^{\frac{1}{2}}/2}^{u_1}dv\int_{\pi^{\frac{1}{2}}/2}^{v}K_n^2H_n^2du$$

$$=2\int_0^x e^{-t^2}dt\int_0^t K_n^2H_n^2e^{-t_1^2}dt_1=I_n.$$

(6.4) is thus established.

As for the left side of (6 4), as in (4.7) $F_n(u)\leqslant u$, and hence since $F_n(u)$ is an increasing sequence it is always convergent and in particular $\left\{\frac{\pi^{\frac{1}{2}}}{2}-F_n\left(\frac{\pi^{\frac{1}{2}}}{2}\right)\right\}=k_n\to k_0$ as $n\to\infty$.

Hence I_n of (6.4) converges. (6.8)

We shall now prove that I_n converges to zero. (6.9)

Let $s_n=\sum_{r-1}^{n}K_r^2H_{r-1}^2$, $s_0=0$;

then $s_n=\left[F_n(u)-\pi^{-\frac{1}{2}}u^2\right]e^{2x^2}\leqslant\left(u-\pi^{-\frac{1}{2}}u^2\right)e^{2x^2}\leqslant Ke^{2x^2}$, (6.10)

where K is the maximum of $u-\pi^{-\frac{1}{2}}u^2$ in $0\leqslant u\leqslant\pi^{\frac{1}{2}}$.

Hence in any finite interval s_n converges boundedly. (6.11)

Now $K_n^2H_n^2 = 2(n+1)K_{n+1}^2H_n^2 = 2(n+1)(s_{n+1}-s_n)$

and $\sum_{r=0}^{n} K_r^2H_r^2 \doteq 2\sum_{r=0}^{n}(r+1)(s_{r+1}-s_r)$

$$= 2\{(n+1)s_{n+1}-s_1-s_2-\ldots-s_n\}.$$

Hence $\dfrac{\sum_{r=0}^{n} K_r^2H_r^2}{n+1} = 2\left\{s_{n+1}-\dfrac{s_1+s_2+\ldots+s_n}{n+1}\right\}$

$$\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ since } s_n \text{ converges,} \tag{6.12}$$

and
$$\frac{\Sigma K_r^2H_r^2}{n+1} \leqslant 2\, s_{n+1} \leqslant Ke^{2x^2}. \tag{6.13}$$

Hence $\dfrac{I_0+I_1+\ldots+I_n}{n+1} = \displaystyle\int_0^x e^{-t^2}dt\int_0^t \frac{\sum_0^n K_r^2H_r^2}{n+1}e^{-u^2}du$

tends to zero by (6.12) and (6.13), for every finite x, as $n \rightarrow \infty$. (6.14)

Since I_n converges, we have $\lim_{n\rightarrow\infty} I_n = 0$. (6.15)

Hence from (6.8) and (6.9)

$$\pi^{-\frac{1}{2}}u^2+e^{-2x^2}\sum_1^\infty K_n^2H_{n-1}^2 = u-k_0. \tag{6.16}$$

Since the left side of (6.16) $\leqslant u$, $k_0 \geqslant 0$; but since the left side $\geqslant 0$ in $0 \leqslant u \leqslant \pi^{\frac{1}{2}}$, $k_0 \leqslant 0$.

Hence $\pi^{-\frac{1}{2}}u^2+e^{-2x^2}\sum_1^\infty K_n^2H_{n-1}^2 = u$, thus establishing (6.3).

We shall now deduce Parseval's theorem for the system $\{K_nH_ne^{-x^2/2}\}$ in $-\infty \leqslant x \leqslant \infty$. Let $f(x)$ be L^2 in $(-\infty, \infty)$ and $f_A(x)=f(x)$ in $-A \leqslant x \leqslant A$ and $f_A(x)=0$ outside $(-A, A)$. Considering $f_A(x)e^{x^2/2}$ as a function of u in $0 \leqslant u \leqslant \pi^{\frac{1}{2}}$, since Parseval's theorem is true for $\{K_nH_n\}$ in $0 \leqslant u \leqslant \pi^{\frac{1}{2}}$, we have

$$\int_0^{\pi^{\frac{1}{2}}} f_A^2(x)e^{x^2}du=\int_{-\infty}^{\infty} f_A^2(x)dx=\sum\left\{\int_0^{\pi^{\frac{1}{2}}} f_A(x)e^{\frac{x^2}{2}}K_nH_n du\right\}^2$$

$$=\sum\left\{\int_{-\infty}^{\infty} f_A(x)K_nH_ne^{-x^2/2}dx\right\}^2. \tag{6.17}$$

(6.17) implies that Parseval's theorem is true for $f_A(x)$ with regard to $\left\{K_nH_ne^{-\frac{x^2}{2}}\right\}$ in $-\infty<x<\infty$. (6.18)

Since $\int_{-\infty}^{\infty}\left\{f(x)-f_A(x)\right\}^2=\int_{-\infty}^{-A}f^2dx+\int_A^{\infty}f^2dx=\varepsilon_A$,

we have by Theorem L_1,

$$\lim_{A\to\infty}\sum\left\{\int_{-\infty}^{\infty} f_AK_nH_ne^{-\frac{x^2}{2}}dx\right\}^2=\sum\left\{\int_{-\infty}^{\infty} fK_nH_ne^{-\frac{x^2}{2}}dx\right\}^2. \tag{6.19}$$

Since by hypothesis, $\int_{-\infty}^{\infty}f_A^2\,dx\to\int_{-\infty}^{\infty}f^2dx$ as $A\to\infty$, by (6.17), and (6.19), we have

$$\int_{-\infty}^{\infty}f^2dx=\sum\left\{\int_{-\infty}^{\infty} fK_nH_ne^{-\frac{x^2}{2}}dx\right\}^2 \tag{6.20}$$

thus establishing Parseval's theorem for $\left\{K_nH_ne^{-\frac{x^2}{2}}\right\}$ in $-\infty<x<\infty$.

7. An alternative proof of (6.3), not so elementary as the above but far shorter, is given below. It is well known* that

$$\frac{e^{-x^2\frac{(1-t)}{1+t}}}{\left\{\pi(1-t^2)\right\}^{\frac{1}{2}}}=e^{-x^2}\sum_0^{\infty}K_n^2H_n^2t^n,\ |t|<1. \tag{7.1}$$

Since the right side consists of positive terms and the integral of the left side in t in (0, 1) exists,

we have
$$\int_0^1\frac{e^{-x^2\frac{(1-t)}{1+t}}}{\pi^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}}dt=\sum e^{-x^2}\frac{K_n^2H_n^2}{n+1}, \tag{7.2}$$

or
$$I=\int_0^1\frac{e^{\frac{-2x^2}{1+t}}}{(1-t^2)^{\frac{1}{2}}}dt=\pi^{\frac{1}{2}}e^{-2x^2}\sum\frac{K_n^2H_n^2}{n+1}=2\pi^{\frac{1}{2}}e^{-2x^2}\sum_0^{\infty}K_{n+1}^2H_n^2. \tag{7.3}$$

* Wiener, *Fourier Integral*, page 60.

Transforming I by $\frac{1-t}{1+t}=u^2$, it becomes $2\int_0^1 e^{-x^2(1+u^2)}\frac{du}{1+u^2}$ (7.4)

and $$\frac{dI}{dx}=-4x\int_0^1 e^{-x^2(1+u^2)}\,du=-4e^{-x^2}\int_0^x e^{-t^2}\,dt.$$

Writing $$u=\int_{-\infty}^x e^{-t^2}dt, \text{ as in } \mathbf{6}$$

we have $$\frac{dI}{du}=-4\left(u-\frac{\pi^{\frac{1}{2}}}{2}\right). \tag{7.5}$$

Hence $I=-2u^2+2\pi^{\frac{1}{2}}u$, since $I=0$ when $u=0$. (7.6)
Hence from (7.3) we have

$$e^{-2x^2}\sum_0^\infty K_{n+1}^2H_n^2=u-\pi^{-\frac{1}{2}}u^2,$$

thus establishing (6.3).

8. *Laguerre polynomials.* As in **6**, we shall establish condition (C) for the system $\left\{\frac{L_n}{n!}\right\}$ where $L_n(x)=e^xD^n(x^ne^{-x})$, considered as functions of u where $u=e^{-x}$ in $0\leqslant u\leqslant 1$.

We have $$\int_0^1 L_nL_m\,du=\int_0^\infty L_nL_me^{-x}dx=0,\ n\neq m \text{ and}$$
$$\int_0^1 L_n^2\,du=\int_0^\infty L_n^2e^{-x}dx=(n!)^2.$$

Hence $\left\{\frac{L_n}{n!}\right\}$ is an $O.N.$ system in $0\leqslant u\leqslant 1$. Condition (C) to be proved in this case will be $\sum\left\{\int_0^u\frac{L_n}{n!}du\right\}^2=u.$ (8.1)

Now $$\frac{d}{dx}(L_n-nL_{n-1})=-nL_{n-1}.$$

Hence $$\frac{d}{dx}(L_n-nL_{n-1})-(L_n-nL_{n-1})=-L_n. \tag{8.2}$$

or $$\frac{d}{dx}\Big(e^{-x}(L_n-nL_{n-1})\Big)=-e^{-x}L_n.$$

Hence $$\int_x^\infty L_ne^{-x}dx=e^{-x}(L_n-nL_{n-1}). \tag{8.3}$$

Since $\int_x^\infty L_n e^{-x} dx = \int_0^u L_n du$, (8.1) becomes

$$e^{-2x}\left\{1+\sum_{n=1}^{\infty}\left(\frac{L_n - nL_{n-1}}{n!}\right)^2\right\} = e^{-x}. \tag{8.4}$$

Writing $\phi_n = \frac{L_n}{n!}$, (8.4) becomes

$$1+\sum_1^\infty (\phi_n - \phi_{n-1})^2 = e^x. \tag{8.5}$$

We shall now prove (8.5). It is well known that

$$e^{-\frac{xt}{1-t}} = (1-t)\Sigma\phi_n t^n = 1+\sum_1^\infty (\phi_n-\phi_{n-1})t^n \text{ for } |t|<1.$$

The function $e^{-\frac{xt}{1-t}} = e^{\frac{x}{2}\left|1+i\cot\frac{u}{2}\right|}$ on the boundary $t = e^{iu}$, $0<u<2\pi$.

Hence it is bounded on $|t|=1$ and is continuous at all points u, except at $u=0$.

Hence for $t=e^{iu}$, $\frac{1}{2\pi}\int_0^{2\pi}\left|e^{-\frac{xt}{1-t}}\right|^2 du = 1+\sum_1^\infty(\phi_n-\phi_{n-1})^2$.

Now $\left|e^{-\frac{xt}{1-t}}\right| = e^{\frac{x}{2}}$ at all u except $u=0$ near which it is bounded on $|t|=1$.

Hence $1+\sum_1^\infty (\phi_n-\phi_{n-1})^2 = e^x$, thus establishing (8.5).

From this as in **6** we can deduce Parseval's theorem for the system $\left\{\phi_n e^{-\frac{x}{2}}\right\}$ in $0 \leqslant x < \infty$.

THE METHOD OF DIFFERENCE EQUATIONS APPLIED TO THE SUMMATION OF CERTAIN SERIES INVOLVING LEGENDRE AND BESSEL FUNCTIONS.

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Introduction. In this paper, certain series, finite as well as infinite, involving Legendre and Bessel functions as well as their allied functions, are summed up by the help of the linear difference equations satisfied by these functions. Some of the results are not new, having been deduced by other writers by special methods. But the methods of this paper are claimed to be quite elegant and elementary, and are capable of being applied to the summation of several series whose terms satisfy certain linear difference equations of the second order. The methods can also possibly be extended so as to be applicable to some other types of linear difference equations.

1. It is well known that the general solution of the second order linear difference equation

$$a_n(z)\phi_{n+2}(z)+b_n(z)\phi_{n+1}(z)+c_n(z)\phi_n(z)=\alpha_n(z) \qquad (1)$$

is given by

$$\phi_n(z)=\lambda(z)\theta_n(z)+\mu(z)\psi_n(z)+h_n(z),$$

where $h_n(z)$ is any particular solution of (1), $\lambda(z)$ and $\mu(z)$ are arbitrary functions independent of n, and $\theta_n(z)$, $\psi_n(z)$ are a pair of linearly independent solutions of the equation obtained by replacing $\alpha_n(z)$ in (1) by zero. It will serve a useful purpose to write down here the solutions of certain particular difference equations which are usually spoken of as the recurrence formulae for the functions of Legendre, Bessel, and allied functions.

(*a*) The functions $P_n(z)$ and $Q_n(z)$ both satisfy the difference equation

$$(n+1)\phi_{n+1}(z)-(2n+1)z\phi_n(z)+n\phi_{n-1}(z)=0. \qquad (2)$$

Hence, any solution of Legendre's differential equation

$$(1-z^2)\frac{d^2u}{dz^2}-2z\frac{du}{dz}+n(n+1)u=0 \qquad (3)$$

satisfies (2). But the converse is obviously not true, since the general solution of (2) is given by

$$\phi_n(z)=f(z)P_n(z)+g(z)Q_n(z), \tag{4}$$

where $f(z)$ and $g(z)$ are arbitrary functions. The general solution of (3) is obtained when $f(z)$ and $g(z)$ reduce to constants.

An important particular solution of (2) is Gauss's polynomial of order $n-1$ given by

$$f_{n-1}(z)=\tfrac{1}{2}P_n(z)\log\frac{z+1}{z-1}-Q_n(z). \tag{5}$$

It will be convenient to call the equation (2) as *Legendre's difference equation.*

(*b*) *The modified (associate) difference equation of Legendre* is similarly

$$(n-m+1)\phi_{n+1}(z)-(2n+1)z\phi_n(z)+(n+m)\phi_{n-1}(z)=0, \tag{6}$$

where m is a fixed positive integer. The general solution is

$$\phi_n(z)=f(z)P_n^m(z)+g(z)Q_n^m(z). \tag{7}$$

If $f(z)$ and $g(z)$ are replaced by constants, (7) gives the general solution of the associated differential equation of Legendre, viz.

$$(1-z^2)\frac{d^2u}{dz^2}-2z\frac{du}{dz}+\left\{n(n+1)-\frac{m^2}{1-z^2}\right\}u=0.$$

(*c*) The equation

$$\phi_{n+1}(z)-\frac{2n}{z}\phi_n(z)+\phi_{n-1}(z)=0, \tag{8}$$

which may be called *Bessel's difference equation*, is known to have each of the following functions as a solution:*

$$J_n(z), Y_n(z), Y^{(n)}(z), H_n^{(1)}(z), H_n^{(2)}(z). \tag{9}$$

It is remarkable that each of the functions in (9) is also a solution of Bessel's differential equation

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(1-\frac{n^2}{z^2}\right)u=0. \tag{10}$$

Hence every solution of (10) also satisfies (8), but the general solution of (8) is

$$\phi_n(z)=f(z)\theta_n(z)+g(z)\psi_n(z),$$

where $\theta_n(z)$ and $\psi_n(z)$ are any two of the functions in (9).

* G. N. Watson, *Treatise on the theory of Bessel Functions.* § 3.6.
Whittaker and Watson, *Modern Analysis.* Chapter XVII.
Ganesh Prasad, *Treatise on Spherical Harmonics.* Chapter VII.

Amongst the solutions of (8), we may note the *cylinder function* which satisfies the additional relation

$$\phi_{n-1}(z)-\phi_{n+1}(z)=2\phi_n'(z).$$

(*d*) The general solution of

$$\phi_{n+1}(z)+\frac{2n}{z}\phi_n(z)-\phi_{n-1}(z)=0 \tag{11}$$

is

$$\phi_n(z)=f(z)I_n(z)+g(z)K_n(z), \tag{12}$$

where $I_n(z)$ and $K_n(z)$ are the *modified* Bessel functions. In particular, the equation (11) is satisfied by any solution of the modified differential equation of Bessel, viz.,

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}-\left(1+\frac{n^2}{z^2}\right)u=0.$$

2. We shall, in passing, write down the solutions of certain difference equations which are reducible to the types given by (*a*), (*b*), (*c*) or (*d*) in 1.

Consider the equation

$$(n+1)\phi_{n+1}(z)-(2n+1)h(z)\phi_n(z)+n\phi_{n-1}(z)=0. \tag{13}$$

Writing $h(Z)$ for z in (2), it becomes

$$(n+1)\phi_{n+1}\left\{h(Z)\right\}-(2n+1)h(Z)\phi_n\left\{h(Z)\right\}+n\phi_{n-1}\left\{h(Z)\right\}=0.$$

Hence if $\phi_n(z)$ is a solution of (2), $\phi_n\left\{h(z)\right\}$ is a solution of (13). The general solution of (13) is therefore

$$\phi_n(z)=f(z)P_n\left\{h(z)\right\}+g(z)Q_n\left\{h(z)\right\}.$$

The solutions of the following difference equations can similarly be deduced from those of the equations (*b*), (*c*), (*d*):

$$(n-m+1)\phi_{n+1}(z)-(2n+1)h(z)\phi_n(z)+(n+m)\phi_{n-1}(z)=0 \tag{14}$$

$$\phi_{n+1}(z)-\frac{2n}{h(z)}\phi_n(z)+\phi_{n-1}(z)=0 \tag{15}$$

$$\phi_{n+1}(z)+\frac{2n}{h(z)}\phi_n(z)-\phi_{n-1}(z)=0. \tag{16}$$

3. *Summation of certain finite series.*

Consider the functional equation

$$\alpha_n\phi_{n+1}(z)+\beta_n h(z)\phi_n(z)+\alpha_{n-1}\phi_{n-1}(z)=0, \tag{17}$$

where α_n and β_n are functions of n. If $\theta_n(z)$ and $\psi_n(z)$ are two solutions of (17), we have

$$\alpha_n\theta_{n+1}(x)+\beta_n h(x)\theta_n(x)+\alpha_{n-1}\theta_{n-1}(x)=0$$

$$\alpha_n\psi_{n+1}(y)+\beta_n h(y)\psi_n(y)+\alpha_{n-1}\psi_{n-1}(y)=0.$$

Multiplying these by $\psi_n(y)$ and $\theta_n(x)$ respectively, and subtracting, we get

$$U_n=\gamma_n-\gamma_{n-1},$$

where $$U_n=\beta_n\theta_n(x)\psi_n(y)$$

and $$\gamma_n=\alpha_n\frac{\theta_{n+1}(x)\psi_n(y)-\theta_n(x)\psi_{n+1}(y)}{h(y)-h(x)}.$$

Since $$\sum_{n=1}^{p}U_n=\gamma_p-\gamma_0,\text{ we have}$$

$$\sum_{n=1}^{p}\beta_n\theta_n(x)\psi_n(y)$$

$$=\frac{\alpha_p\{\theta_{p+1}(x)\psi_p(y)-\theta_p(x)\psi_{p+1}(y)\}-\alpha_0\{\theta_1(x)\psi_0(y)-\theta_0(x)\psi_1(y)\}}{h(y)-h(x)}. \qquad (A)$$

In a similar manner, for the equation

$$\alpha_n\phi_{n+1}(z)+\beta_n h(z)\phi_n(z)-\alpha_{n-1}\phi_{n-1}(z)=0 \qquad (18)$$

we have

$$U_n=\gamma_n+\gamma_{n-1},$$

where U_n and γ_n have the same meanings as before, except that $\theta_n(z)$ and $\psi_n(z)$ are now solutions of (18).

Since $$\sum_{n=1}^{p}(-1)^{n-1}U_n=\gamma_0+(-1)^{p-1}\gamma_p,$$

we get

$$\sum_{n=1}^{p}(-1)^{n-1}\beta_n\theta_n(x)\psi_n(y)$$

$$=\frac{1}{h(y)-h(x)}\Big[\alpha_0\{\theta_1(x)\psi_0(y)-\theta_0(x)\psi_1(y)\}$$

$$+(-1)^{p-1}\alpha_p\{\theta_{p+1}(x)\psi_p(y)-\theta_p(x)\psi_{p+1}(y)\}\Big]. \qquad (B)$$

4. We shall now deduce from (A) and (B) a number of results by taking the difference equations considered in **1.**

The equation

$$(n+1)\phi_{n+1}(z)-(2n+1)z\phi_n(z)+n\phi_{n-1}(z)=0 \qquad (2)$$

is a particular case of (17), wherein

$$\alpha_n=n+1,\ \beta_n=2n+1 \text{ and } h(z)=-z.$$

The functions $P_n(z)$, $Q_n(z)$ and $f_{n-1}(z)$ are solutions of (2). Hence, taking $\theta_n(x)=P_n(x)$, $\psi_n(y)=P_n(y)$, and using the relation

$$P_1(x)P_0(y)-P_0(x)P_1(y)=(x-y)P_0(x)P_0(y),$$

we get Christoffel's result

$$\sum_{n=0}^{p}(2n+1)P_n(x)P_n(y)=(n+1)\frac{P_{p+1}(x)P_p(y)-P_p(x)P_{p+1}(y)}{x-y}. \qquad \text{(i)}$$

Taking $\theta_n(x)=P_n(x)$, $\psi_n(y)=Q_n(y)$, and using

$$P_1(x)Q_0(y)-P_0(x)Q_1(y)=1+(x-y)P_0(x)Q_0(y)$$

we get Heine's result

$$\sum_{n=0}^{p}(2n+1)P_n(x)Q_n(y)$$

$$=\frac{1}{y-x}-(p+1)\frac{P_{p+1}(x)Q_p(y)-P_p(x)Q_{p+1}(y)}{y-x}. \qquad \text{(ii)}$$

Taking $\theta_n(x)=Q_n(x)$, $\psi_n(y)=Q_n(y)$, and using

$$Q_1(x)Q_0(y)-Q_0(x)Q_1(y)$$

$$=\tfrac{1}{2}\log\frac{(x+1)(y-1)}{(x-1)(y+1)}+(x-y)Q_0(x)Q_0(y)$$

we get Frobenius's result

$$\sum_{n=0}^{p}(2n+1)Q_n(x)Q_n(y)=\frac{1}{2(y-x)}\log\frac{(x+1)(y-1)}{(x-1)(y+1)}$$

$$-(p+1)\frac{Q_{p+1}(x)Q_p(y)-Q_p(x)Q_{p+1}(y)}{y-x}. \qquad \text{(iii)}$$

We have similarly

$$\sum_{n=0}^{p}(2n+1)P_n(x)f_{n-1}(y)$$

$$=\frac{1}{x-y}+(p+1)\frac{P_{p+1}(x)\,f_{p-1}(y)-P_p(x)f_p(y)}{x-y}. \qquad \text{(iv)}$$

$$\sum_{n=0}^{p} (2n+1)Q_n(x)f_{n-1}(y) = \frac{1}{2(x-y)} \log \frac{x+1}{x-1}$$
$$+(p+1)\frac{Q_{p+1}(x)f_{p-1}(y)-Q_p(x)f_p(y)}{x-y}. \quad \text{(v)}$$

$$\sum_{n=1}^{p} (2n+1)f_{n-1}(x)f_{n-1}(y)$$
$$= (p+1)\frac{f_p(x)f_{p-1}(y)-f_{p-1}(x)f_p(y)}{x-y}, \quad \text{(vi)}$$

it being understood that $f_0(z) = 1$, $f_{-1}(z) = 0$.

The results (iv), (v) and (vi) are of course deducible from (i), (ii) and (iii) by using the relation (5).

By evaluating the limiting values of the right sides of the equations (i)—(vi) when $x-y$ tends to zero, we shall be obtaining the sums of the six *finite* series

$$\sum_{n=0}^{p} (2n+1)[P_n(x)]^2, \ \sum_{n=0}^{p} (2n+1)P_n(x)Q_n(x),$$

$$\sum_{n=0}^{p} (2n+1)[Q_n(x)]^2, \sum_{n=0}^{p} (2n+1)P_n(x)f_{n-1}(x),$$

$$\sum_{n=0}^{p} (2n+1)Q_n(x)f_{n-1}(x), \ \sum_{n=0}^{p} (2n+1)[f_{n-1}(x)]^2.$$

We next take the equation

$$\phi_{n+1}(z) - \frac{2n}{z}\phi_n(z)+\phi_{n-1}(z) = 0 \quad (8)$$

whose general solution is given in **1**(c). We have here $\alpha_n = 1$, $\beta_n = n$, $h(z) = -2/z$. Taking for $\theta_n(x)$, $\psi_n(y)$ any of the functions J_n, Y_n, $Y^{(n)}$, $H_n^{(1)}$ and $H_n^{(2)}$, we get the following results:

(a). $\sum_{n=1}^{p} n J_n(x)J_n(y)$

$$= \frac{xy}{2(y-x)}[J_{p+1}(x)J_p(y)-J_p(x)J_{p+1}(y)$$
$$-J_1(x)J_0(y)+J_0(x)J_1(y)]$$

(b)—(e). Similar equations obtained by changing J_r in (a) to Y_r, $Y^{(r)}$, $H_n^{(1)}$, $H_n^{(2)}$ respectively.

(f). $\sum_{1}^{p} nJ_n(x)Y_n(y) = \frac{xy}{2(y-x)}[J_{p+1}(x)Y_p(y)-J_p(x)Y_{p+1}(y)$
$$-J_1(x)Y_0(y)+J_0(x)Y_1(y)],$$

and similar equations by taking any two of the five functions.

The sums of the finite series $\sum_1^p n\{J_n(x)\}^2$, etc. would be obtained by determining the limiting values of the above expressions as $x-y$ tends to zero.

Considering next the equation

$$\phi_{n+1}(z)+\frac{2n}{z}\phi_n(z)-\phi_{n-1}(z)=0 \tag{11}$$

which is of the type given by (18), we get the following:

(α) $\sum_1^p(-1)^{n-1} n\, I_n(x)I_n(y)$

$$=\frac{xy}{2(x-y)}\Big[\{I_1(x)I_0(y)-I_0(x)I_1(y)\}$$
$$+(-1)^{p-1}\{I_{p+1}(x)I_p(y)-I_p(x)I_{p+1}(y)\}\Big].$$

(β) A similar result for $\sum_1^p(-1)^{n-1}nK_n(x)K_n(y)$.

(γ) $\sum_{n=1}^p(-1)^{n-1}nI_n(x)K_n(y)$

$$=\frac{xy}{2(x-y)}\Big[\{I_1(x)K_0(y)-I_0(x)K_1(y)\}$$
$$+(-1)^{p-1}\{I_{p+1}(x)K_p(y)-I_p(x)K_{p+1}(y)\}\Big].$$

5. *Summation of a certain infinite series.* We shall now consider a method of finding the sum (when it exists) of the infinite series

$$V=\sum_{n=0}^{\infty}t^n\phi_n(z), \tag{19}$$

where $\phi_n(z)$ is a solution of the difference equation

$$(n+1)\phi_{n+1}(z)-(2n+1)z\phi_n(z)+n\phi_{n-1}(z)=0.$$

Multiplying by t^n, and summing with respect to n, we have

$$\sum_{n=1}^{\infty}(n+1)t^n\phi_{n+1}(z)-2zt\sum_1^{\infty}nt^{n-1}\phi_n(z)-z\sum_1^{\infty}t^n\phi_n(z)$$
$$+\sum_1^{\infty}(n-1)\,t^n\phi_{n-1}(z)+\sum_0^{\infty}t^n\phi_{n-1}(z)=0. \tag{20}$$

Now, assuming that (19) is uniformly convergent with respect to the variable t, we get on differentiating term by term

$$\frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} nt^{n-1}\phi_n(z). \tag{21}$$

Also we have evidently

$$\left.\begin{aligned} &\sum_{1}^{\infty}(n+1)t^n\phi_{n+1}(z) = \frac{\partial V}{\partial t} - \phi_1(z) \\ &\sum_{1}^{\infty} t^n\phi_n(z) = V - \phi_0(z) \\ &\sum_{1}^{\infty}(n-1)t^n\phi_{n-1}(z) = t^2\frac{\partial V}{\partial t} \\ &\sum_{1}^{\infty} t^n\phi_{n-1}(z) = t\,V. \end{aligned}\right\} \tag{22}$$

Using (22) in (20), we get

$$\left\{\frac{\partial V}{\partial t} - \phi_1(z)\right\} - 2zt\frac{\partial V}{\partial t} - z\left\{V - \phi_0(z)\right\} + t^2\frac{\partial V}{\partial t} + tV = 0,$$

or

$$(1-2zt+t^2)\frac{\partial V}{\partial t} - (z-t)V = \phi_1(z) - z\phi_0(z). \tag{23}$$

Hence, putting $\xi = 1-2zt+t^2$, we can write

$$\xi^{\frac{1}{2}}\frac{\partial}{\partial t}(\xi^{\frac{1}{2}}V) = \phi_1(z) - z\phi_0(z).$$

Integrating, we obtain

$$\xi^{\frac{1}{2}}V = \left\{\phi_1(z) - z\phi_0(z)\right\}\int_0^t \xi^{-\frac{1}{2}}dt + \psi(z). \tag{24}$$

To find $\psi(z)$, we put $t=0$ in (24) and observe that V and ξ then reduce to $\phi_0(z)$ and 1 respectively. Hence $\psi(z) = \phi_0(z)$. Therefore

$$\sum_{n=0}^{\infty} t^n\phi_n(z) = \frac{\phi_1(z) - z\phi_0(z)}{(1-2zt+t^2)^{\frac{1}{2}}}\int_0^t \frac{dt}{(1-2zt+t^2)^{\frac{1}{2}}} + \frac{\phi_0(z)}{(1-2zt+t^2)^{\frac{1}{2}}}. \tag{25}$$

The integral is readily evaluated in terms of the inverse hyperbolic cosine.

The validity of the result (25) of course depends on the condition that (21) should hold. This question will be scrutinised in the next section. We write down here the following corollaries:—

(i) Put $\phi_n(z) = P_n(z)$, $\phi_0(z) = 1$, $\phi_1(z) = z$. We get the familiar result

$$\sum_{n=0}^{\infty} t^n P_n(z) = (1-2zt+t^2)^{-\frac{1}{2}}.$$

(ii) Put $\phi_n(z) = Q_n(z)$, $\phi_0(z) = \frac{1}{2}\log\frac{z+1}{z-1}$,

$$\phi_1(z) = \tfrac{1}{2} z \log\frac{z+1}{z-1} - 1.$$

Then,*

$$\sum_{n=0}^{\infty} t^n Q_n(z) = \frac{1}{2(1-2zt+t^2)^{\frac{1}{2}}}\log\frac{z+1}{z-1} - \frac{1}{(1-2zt+t^2)^{\frac{1}{2}}}\int_0^t \frac{dt}{(1-2tz+t^2)^{\frac{1}{2}}}.$$

(iii) Put $\phi_n(z) = f_{n-1}(z)$ where $f_{n-1}(z)$ is Gauss's polynomial defined by (5); $f_{-1}(z) = 0$, $f_0(z) = 1$. Then

$$\sum_{0}^{\infty} t^n f_{n-1}(z) = \frac{1}{(1-2zt+t^2)^{\frac{1}{2}}}\int_0^t \frac{dt}{(1-2zt+t^2)^{\frac{1}{2}}},$$

which could of course be deduced from (i) and (ii).

6. We shall now look upon t as the variable, and z as a parameter. We write

$$V = \sum_{n=0}^{\infty} a_n t^n, \tag{26}$$

where $a_n = \phi_n(z)$. The recurrence formula satisfied by $\phi_n(z)$ may then be written

* Whittaker and Watson, *loc. cit.* (1935) p. 321. Throughout this paper $Q_n(z)$ is taken as the ordinary Legendrian function of the second kind definable for $|z| > 1$. When however $Q_n(z)$ is defined for $|z| < 1$, many of the results of this paper require slight modifications.

$$a_{n+2} = p_n a_{n+1} + q_n a_n,$$

where $$p_n = \frac{2n+3}{n+2} z \text{ and } q_n = -\frac{n+1}{n+2}.$$

We shall now use a theorem* due to van Vleck, viz:—*If* $p+1$ *successive coefficients of a power series* $\sum_{n=0}^{\infty} a_n t^n$ *conform to a recurrence relation of the type*

$$a_{n+p} = \beta_{0,n} a_n + \beta_{1,n} a_{n+1} + \ldots + \beta_{p-1,n} a_{n+p-1},$$

where p is a fixed positive integer, and n can take any positive integral value, the radius of convergence of the series is equal to the smallest of the moduli of the roots of the equation

$$1 - c_{p-1}k - c_{p-2}k^2 - \ldots - c_1 k^{p-1} - c_0 k^p = 0,$$

where $$c_r = \lim_{n\to\infty} \beta_{r,n} \; (r = 0, 1, \ldots, p).$$

In the present case, $\lim p_n = 2z$, $\lim q_n = -1$. Therefore the radius of convergence of the series (26) is equal to the smaller of the moduli of the two roots of

$$k^2 - 2kz + 1 = 0.$$

Hence, if $$z_1 = z + (z^2-1)^{\frac{1}{2}}, \; z_2 = z - (z^2-1)^{\frac{1}{2}},$$
the series (26) is absolutely and uniformly convergent for all values of t such that

$$|t| < |z_1|, \; |z_2|. \tag{27}$$

The investigation of **5** is then valid under this condition. This condition incidentally implies that $|t| < 1$.

Let us now suppose that t has a pre-assigned value (real or complex) of modulus less than unity, and proceed to investigate the region in the Argand diagram within which the point z has to lie in order to satisfy the conditions (27).

7. The identity

$$\left(a + \frac{1}{a}\right) - \left(b + \frac{1}{b}\right) = (a-b)\left(1 - \frac{1}{ab}\right)$$

gives

$$a + \frac{1}{a} > b + \frac{1}{b}, \tag{28}$$

* Edward B. Van Vleck, 'Linear criteria for the determination of the radius of convergence of a power series', *Trans. Amer. Math. Soc*, 1 (1900), 293-309.

provided that a and b are positive quantities such that $a < b$; $ab < 1$. If now $|z_1| < |z_2|$, putting $a = |t|$, $b = |z_1|$, $b^{-1} = |z_2|$, we get from (27), $a < b$; $ab < 1$. Hence by (28),

$$|z_1| + |z_2| = |z_1| + |z_1^{-1}| < |t| + |t^{-1}|. \tag{29}$$

If on the other hand $|z_1| > |z_2|$, we take $a = |t|$, $b = |z_2|$, and then we get (29) as before. The case $|z_1| = |z_2|$ merely leads to $|t| < 1$, by (27).

We conclude that the condition for uniform and absolute convergence of the series (26) is that

$$|z + (z^2-1)^{\frac{1}{2}}| + |z - (z^2-1)^{\frac{1}{2}}| < |t| + |t^{-1}| \tag{30}$$

unless $|z \pm (z^2-1)^{\frac{1}{2}}| = 1$, in which case $|t| < 1$ is the condition in question.

The relation (30) can also be written

$$|z+1| + |z-1| < |t| + |t^{-1}|, \tag{31}$$

since the left side expressions in (30) and (31) are equal. The result could be put in geometrical language. If we construct the ellipse whose real foci are the points ± 1, and whose major axis is $|t| + |t^{-1}|$, the ellipse is evidently real since $|t| + |t^{-1}| > 2$. The point z satisfying (31) lies inside this ellipse. The result (25) is now written in its full form:

If $\phi_n(z)$ be any analytic solution of the difference equation

$$(n+1)\phi_{n+1}(z) - (2n+1)z\phi_n(z) + n\phi_{n-1}(z) = 0,$$

and if t be a preassigned quantity of modulus less than unity, then the power series $\sum_{n=0}^{\infty} t^n \phi_n(z)$ is absolutely and uniformly convergent to the sum

$$\frac{\phi_1(z) - z\,\phi_0(z)}{(1-2zt+t^2)^{\frac{1}{2}}} \int_0^t \frac{dt}{(1-2zt+t^2)^{\frac{1}{2}}} + \frac{\phi_0(z)}{(1-2zt+t^2)^{\frac{1}{2}}},$$

provided that the point z is an interior point of the ellipse, whose real foci are at ± 1, and whose major axis is of length $|t| + |t^{-1}|$.

If we suppose that $\phi_n(z)$ can possess singular points inside the above domain, the set of such points should be excluded from the above domain of convergence. Thus if $\phi_n(z) = Q_n(z)$, the portion of the elliptic domain which has to be left out of consideration consists either of the part of the unit circle overlapping the ellipse, or the rectilinear segment having ± 1 as end-points, accor

ing as the one or the other of the two following definitions of $Q_n(z)$ is adopted:—

(1) $$Q_n(z) = \frac{1}{2z^{n+1}}\left[I_{n,n} + \frac{I_{n+2,n}}{z^2} + \frac{I_{n+4,n}}{z^4} + \dots + \frac{I_{n+2k,n}}{z^{2k}} + \dots \text{ to } \infty\right],$$

where

$$I_{m,n} = \int_{-1}^{+1} v^m P_n(v)\,dv.$$

(2) $$Q_n(z) = \tfrac{1}{2}\int_{-1}^{+1} \frac{P_n(v)}{z-v}\,dv.$$

ON SOME NEW INTEGRAL RELATIONS BETWEEN BESSEL AND LEGENDRE FUNCTIONS OF UNRESTRICTED DEGREE

By D. P. BANERJEE, *Mymensingh* (*Bengal*).

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S. C. Mitra* has found certain integral relations between Bessel and Legendre functions of positive integral degree. Here I shall find the integral relations between Bessel and Legendre functions of unrestricted degree and express the sums of certain series as definite integrals.

The Bessel function† $J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\Pi(s)\Pi(n+s)}\left(\frac{x}{2}\right)^{n+2s}$.

The Legendre function‡

$$P_n(x)=F\left(n+1,-n,\,1,\,\frac{1-x}{2}\right)\ \text{if}\ |1-x|<2.$$

We know§

$$J_n(p)\cos p=\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})}{\Pi(2s)\ \Pi(2n+2s)}(2p)^{n+2s}$$

$$J_n(p)\sin p=\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s+\frac{1}{2})}{\Pi(2s+1)\ \Pi(2n+2s+1)}(2p)^{n+2s+1}.$$

Hence $J_n(p)\cos p+J_{n+1}(p)\sin p$

$$=\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})}{\Pi(2s)\ \Pi(2n+2s)}(2p)^{n+2s}$$

$$+\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s+\frac{3}{2})}{\Pi(2s+1)\ \Pi(2n+2s+3)}(2p)^{n+2s+2}$$

* S. C. Mitra, *Math. Zeit.* Band., 41, Heft 5, p. 681.
† Gray, *A Treatise on Bessel functions* (1922), p. 14.
‡ G. Prasad, *Spherical Harmonics* (1932), p. 7.
§ Gray, *loc. cit.*, p. 18.

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$$=\frac{1}{\pi^{\frac{1}{2}}}\left[\frac{\Pi(n-\frac{1}{2})}{\Pi(2n)}(2p)^n-\frac{\Pi(n+\frac{3}{2})(2n+1)}{\Pi(2)\,\Pi(2n+3)}(2p)^{n+2}\right.$$
$$\left.-\ldots+\frac{(-1)^s\Pi(n+2s-\frac{1}{2})(2n+1)}{\Pi(2s)\,\Pi(2n+2s+1)}(2p)^{n+2s}+\ldots\right]$$
$$=\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})(2n+1)}{\Pi(2s)\,\Pi(2n+2s+1)}(2p)^{n+2s},$$

by adding the $(s-1)$th term of the second series with the sth of the first series.

Hence
$$\frac{1}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})}{\Pi(2s)\,\Pi(2n+2s+1)}(2p)^{n+2s}$$
$$=\frac{1}{2n+1}[J_n(p)\cos p+J_{n+1}(p)\sin p].$$

Again*
$$\int_0^1 P_{n+\frac{1}{2}}(x)\,x^{n+2s-\frac{1}{2}}dx$$
$$=\frac{2^{n+\frac{1}{2}}\Pi(n+2s-\frac{1}{2})}{\Pi(2n+2s+1)}\,\frac{\Pi(n+s)}{\Pi(s-\frac{1}{2})},\ \text{if } R(n)>-\tfrac{1}{2}.$$

Hence if $R(n)>-\frac{1}{2}$,
$$\int_0^1 P_{n+\frac{1}{2}}(x)J_n(2px)\,x^{-\frac{1}{2}}dx$$
$$=\sum_{s=0}^{\infty}\frac{(-1)^s p^{n+2s}}{\Pi(n+s)\,\Pi(s)}\,\frac{2^{n+\frac{1}{2}}\Pi(n+2s-\frac{1}{2})\,\Pi(n+s)}{\Pi(2n+2s+1)\,\Pi(s-\frac{1}{2})}$$
$$=2^{\frac{1}{2}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})}{\Pi(2n+2s+1)}\,\frac{(2p)^{n+2s}}{2^{2s}\,\Pi(s)\Pi(s-\frac{1}{2})},$$

by expanding $J_n(2px)$ and integrating term by term, which can be justified.

Again † $\qquad \Pi(2s)=\pi^{-\frac{1}{2}}\Pi(s)\,\Pi(s-\frac{1}{2})2^{2s}.$

Hence
$$\int_0^1 P_{n+\frac{1}{2}}(x)J_n(2px)\,x^{-\frac{1}{2}}dx$$
$$=\frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s\Pi(n+2s-\frac{1}{2})}{\Pi(2n+2s+1)\,\Pi(2s)}(2p)^{n+2s}$$

* Barnes, *Quarterly Journal of Mathematics*, Vol. 39, p. 182.
† Gray, *loc. cit.*, p. 255.

$$= \frac{2^{\frac{1}{2}}}{2n+1}[J_n(p) \cos p + J_{n+1}(p) \sin p]. \qquad (1)$$

Differentiating both the sides with respect to p and using the recurrence formulae satisfied by $J_n(p)$*, we get

$$\int_0^1 P_{n+\frac{1}{2}}(x) J'_n(2px) x^{\frac{1}{2}} dx$$

$$= \frac{1}{2^{\frac{1}{2}}(2n+1)}\left[\frac{nJ_n(p) \cos p - (n+1)J_{n+1}(p) \sin p}{p}\right]. \qquad (2)$$

Putting ip for p we have from (1) and (2)

$$\int_0^1 P_{n+\frac{1}{2}}(x) I_n(2px)\, x^{-\frac{1}{2}} dx$$

$$= \frac{2^{\frac{1}{2}}}{2n+1}\left[I_n(p) \cosh p - I_{n+1}(p) \sinh p\right]; \qquad (3)$$

$$\int_0^1 P_{n+\frac{1}{2}}(x) I_n'(2px) x^{\frac{1}{2}} dx$$

$$= \frac{1}{2^{\frac{1}{2}}(2n+1)}\left[\frac{nI_n(p) \cosh p + (n+1)I_{n+1}(p) \sinh p}{p}\right]. \qquad (4)$$

We know†

$$P_p(1-2x^2)P_q(1-2x^2) =$$

$$\sum_{r=0}^{q} \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)P_{p+q-2r}(1-2x^2) \qquad (A)$$

where

$$q \leqslant p, \qquad 0 \leqslant u \leqslant 1,\ A_r = \frac{2^r(\frac{1}{2})_r}{r!},\ A_{-r} \doteq \frac{(-1)^r 2^r}{(\frac{1}{2})_r r!},$$

and p, q are positive integers.

Also‡ $$\int_0^1 P_n(1-2y^2) J_0(yz) y dy = \frac{J_{2n+1}(z)}{z}$$

$$\int_0^1 P_n(1-2y^2) \sin 2yz\, dy = \frac{\pi}{2} J^2_{n+\frac{1}{2}}(z),$$

$$\int_0^1 P_n(1-2y)^2 \cos 2yz\, dy = \frac{\pi}{2}(-1)^n J_{n+\frac{1}{2}}(z) J_{-(n+\frac{1}{2})}(z).$$

* Gray, *loc. cit.*, p. 15.

† Adams, *Proc. Royal. Soc.* XXVII.

‡ S. C. Mitra, *loc. cit.*

And*

$$I_{n+\frac{1}{2}}(x)K_{n+\frac{1}{2}}(x)=\int_0^1 e^{-2yx}P_n(1-2y^2)\,dy$$

where n is a positive integer.

Applying these results to (A), we have

$$\int_0^1 P_p(1-2y^2)P_q(1-2y^2)J_0(yz)\,y\,dy$$

$$=\sum_{r=0}^{q}\frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)\frac{J_{2p+2q-4r+1}(z)}{z}; \qquad (5)$$

$$\int_0^1 P_p(1-2y^2)P_q(1-2y^2)\sin 2yz\,dy$$

$$=\frac{\pi}{2}\sum_{r=0}^{q}\frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)J^2_{p+q-2r+\frac{1}{2}}(z); \qquad (6)$$

$$\int_0^1 P_p(1-2y^2)P_q(1-2y^2)\cos 2yz\,dy$$

$$=\frac{\pi}{2}\sum_{r=0}^{q}\frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)\times$$

$$(-1)^{p+q}J_{p+q-2r+\frac{1}{2}}(z)J_{-(p+q-2r+\frac{1}{2})}(z). \qquad (7)$$

$$\int_0^1 P_p(1-2y^2)P_p(1-2y^2)e^{-2yz}\,dy$$

$$=\sum_{r=0}^{q}\frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)\times$$

$$I_{p+q-2r+\frac{1}{2}}(z)K_{p+q-2r+\frac{1}{2}}(z). \qquad (8)$$

My sincere thanks are due to the referee for kind criticisms and corrections.

* S. C. Mitra, *Bull. Calcutta Math. Soc.*, 25, p. 92.

NOTE ON THE ADDITION THEOREM OF PARABOLIC CYLINDER FUNCTIONS

By S. C. DHAR, *Nagpur University*

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In a recent paper*, Mr. Hari Shanker has given an addition theorem which can be obtained as a sort of generalization of the result obtained by me in my paper in the *Jour. London Math. Soc.* 10, (1935), 171-75. In this short note I shall show how to obtain it.

We know† that when $R(n) > 0$,

$$e^{-\frac{1}{4}z^2} D_n(z) = e^{\frac{-in\pi}{2}} 2^{n+1} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2u^2+2iuz} u^n du.$$

Hence according to the method shown in my paper quoted above, we can obtain

$$D_n(X)D_m(x) = \sum_{r=0}^{\infty} A_{n+m-r} D_{n+m-r}(X \cos\alpha + x \sin\alpha) D_r(X \sin\alpha - x\cos\alpha), \quad (1)$$

where X and x are any two arguments and $R(m) > 0$, $R(n) > 0$, and where

$$A_{n+m-r} = \sin^{m+r}\alpha \cos^{n-r}\alpha \frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)} {}_2F_1(-r, -m; n-r+1; -\cot^2\alpha).$$

In (1) we make the substitution

$$X\cos\alpha + x\sin\alpha = y, \quad X\sin\alpha - x\cos\alpha = z,$$

and obtain

$$D_n(y\cos\alpha + z\sin\alpha) D_m(y\sin\alpha - z\cos\alpha) = \sum_{r=0}^{\infty} A_{n+m-r} D_{n+m-r}(y) D_r(z). \quad (2)$$

* Hari Shanker: *Jour. Ind. Math. Soc.* Vol. III (2), 1939, pp, 226-230.

† Hari Shanker; *Proc. Ben, Math, Soc.* Vol. VI (1924).

Now, we make $m=0$, and obtain

$$D_n(y\cos\alpha+z\sin\alpha)=$$

$$e^{\frac{1}{4}(y\sin\alpha-z\cos\alpha)^2}\Gamma(n+1)\sum_{r=0}^{\infty}\sin^r\alpha\cos^{n-r}\alpha\,\frac{D_{n-r}(y)}{\Gamma(n-r+1)}\frac{D_r(z)}{\Gamma(r+1)}. \quad (3)$$

Now, we put $y=\rho y_1$, $z=\rho z_1$, and write $a=\rho\cos\alpha$, $b=\rho\sin\alpha$ and obtain on simplification

$$D_n(ay_1+bz_1)=$$

$$e^{\frac{1}{4}(by_1-az_1)^2}\Gamma(n+1)\sum_{r=0}^{\infty}\left(\frac{b}{\rho}\right)^r\left(\frac{a}{\rho}\right)^{n-r}\frac{D_{n-r}(\rho y_1)}{\Gamma(n-r+1)}\frac{D_r(\rho z_1)}{\Gamma(r+1)},$$

which reduces, on simplification and removing the suffixes, to

†$D_n(ay+bz)=$

$$e^{\frac{1}{4}(by-az)^2}\Gamma(n+1)\left(\frac{a}{\rho}\right)^n\sum_{r=0}^{\infty}\frac{D_{n-r}(\rho y)}{\Gamma(n-r+1)}\frac{D_r(\rho z)}{\Gamma(r+1)}\left(\frac{b}{a}\right)^r \quad (4)$$

where $\rho^2=a^2+b^2$.

For the same substitution, (2) will reduce to

$$D_n(ay+bz)D_m(by-az)=\sum_{r=0}^{\infty}A_{m+n-r}D_{n+m-r}(\rho y)D_r(\rho z), \quad (5)$$

where

$$A_{m+n-r}=\left(\frac{b}{\rho}\right)^{m+r}\left(\frac{a}{\rho}\right)^{n-r}\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}$$
$$\times{}_2F_1\left(-r,-m;n-r+1;-\frac{a^2}{b^2}\right). \quad (6)$$

* Dhar: *loc. cit.* p, 173

† Hari Shanker: *loc. cit.* p. 228.

ON THE ROOTS OF A CONTINUOUS NON-DIFFERENTIABLE FUNCTION

By S. MINAKSHISUNDARAM.

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Let $f(x)$ be a continuous non-differentiable function defined in the interval $(0, 1)$, whose upper and lower bounds are l and u, say. If $l \leqslant \alpha \leqslant u$, we denote by $S(\alpha)$, the set of points x in $(0, 1)$ for which $f(x) = \alpha$. It is known that $S(\alpha)$ is closed and non-dense.* We now divide the interval (l, u) into three distinct sets of points A, B and C, so that $A+B+C=(l, u)$, and

(a) $A =$ the set of points α for which $S(\alpha)$ is of positive measure

(b) $B =$ the set of points α for which $S(\alpha)$ is non-enumerable but of measure zero

(c) $C =$ the set of points α for which $S(\alpha)$ is enumerable. We have then the following main theorem of this note:

THEOREM. *For any continuous non-differentiable function* $f(x)$,

(1) *A is at most enumerable,*

(2) *B has the measure of the interval* (l, u),

(3) *C is of measure zero at most.*

In other words a continuous non-differentiable function takes almost every value more than enumerably infinitely often. It is known that in the case of Singh's function A and C are both empty.† In the case of Weierstrass's non-differentiable function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos b^n x\pi$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + \frac{3\pi}{2}(1-a)$,

* R. Vaidyanathaswamy: On continuous functions of a real variable: *Proc. Ind. Acad. Sci.*, **9** (1919), 67-71, Theorem 2.

† A. N. Singh: *The theory and Construction of Non-Differentiable Functions*, p. 89.

I show that $S(l)$ and $S(u)$ are enumerable, so that C is not empty. Also it is shown that C can include only the proper maxima and minima, so that C is at most enumerable.

We now proceed to the proof of the main theorem. That A is at most enumerable has been already observed by Dr. R. Vaidyanathaswamy.* In fact if $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_i > \ldots \to 0$, the set A_i of points α for which the measure of $S(\alpha) > \varepsilon_i$ is finite, so that $A = \sum_{i=1}^{\infty} A_i$ is at most enumerable. It remains only to prove (3) as then (2) will follow easily. Now (3) is a consequence of a lemma of Saks†:

LEMMA. *Suppose that F is continuous, and that E is a set at no point of which the function F has a derivative (finite or infinite). Suppose further that each point x of E is an isolated point of the set $E_t[F(t) = F(x)]$. Then $|E| = |F(E)| = 0$.*

Here $|E|$ denotes the measure of E and $F(E)$ is the set of the values of F over E. Now to every point α of C, we associate a point $x = x_\alpha$ so that

(i) $f(x_\alpha) = \alpha$

(ii) x_α is an isolated point of the set $S(\alpha)$. The choice (ii) is always possible, as $S(\alpha)$, being closed and enumerable, must contain isolated points. We shall denote by X the set of points x_α so chosen as to satisfy (i) and (ii). Now by the lemma, as $f(x)$ is non-differentiable,

$$|f(X)| = |X| = 0.$$

But

$$C = f(X)$$

Hence $|C| = 0$. This proves the theorem.

For Weierstrass's non-differentiable function $W(x)$, C is at most enumerable; for, if the set $S(\alpha)$ were to contain an isolated point x_α, x_α is necessarily a cusp§. And the cusps are enumerable. *But C is certainly not empty,* for $S(l)$ and $S(u)$ are enumerable. In fact the function

$$\sum_{n=0}^{\infty} a^n (\cos b^n x\pi + 1) = W(x) - l$$

* *loc. cit.*, Theorem 3.

† S. Saks: *Theory of the Integral*, p. 277.

§ C. N. Srinivasiengar: On the zeros of Weierstrass's Non-Differentiable Function. *Jour. Ind. Math. Soc.*, (New Series) 3 (1938), p. 117, line 17, and Theorem 2, p. 116.

has its least value zero. As every term of the series on the left side is non-negative, the least value zero is attained, only when every term vanishes. Let Z_n denote the set of zeros of $1+a^n \cos b^n \pi x$, which is enumerable. Then the set of zeros of $\Sigma a^n (\cos b^n \pi x + 1)$ is

$$\prod_{n=1}^{\infty} Z_n = S(l)$$

which clearly is also enumerable. We can similarly show that $S(u)$ is enumerable, as $W(x)$ takes the value u, only when every term takes its corresponding greatest value.

The observation, just now made, that the only isolated points of $S(\alpha)$ are the cusps, enables us to prove that *except for an enumerable set of values* α, *which are proper maxima and minima of* $W(x)$, $S(\alpha)$ *is perfect.* Because $S(\alpha)$, being closed and non-dense, is the sum of an enumerable set and a perfect set, and if $S(\alpha)$ is not perfect, it will contain isolated points, which is possible only if α is a proper maximum or a minimum. A similar remark applies to any continuous non-differentiable function $f(x)$. In general $S(\alpha) = P_\alpha + I_\alpha$, where P_α is a *perfect* set and I_α is composed of *isolated* points. We can now apply the lemma for the α's for which I_α is not empty and conclude that *except possibly for a set of values* α *of measure zero,* $S(\alpha)$ *is a non-dense perfect set.* Again we could take $E = \sum_{l \leqslant \alpha \leqslant u} I_\alpha$ and apply the lemma. We then observe that *the isolated roots of the equation*

$$f(x) = \alpha, \qquad l \leqslant \alpha \leqslant u$$

form a set of measure zero.

ON INTEGRAL REPRESENTATION OF WEBER'S PARABOLIC CYLINDER FUNCTION AND ITS EXPANSION INTO AN INFINITE SERIES

By HARI SHANKER, *Anglo-Arabic College, Delhi.*

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1. In my previous paper* I have obtained an expression for the parabolic cylinder function $D_n(z)$ in the form

$$e^{-\frac{1}{4}z^2} D_n(z) = \sum_{r=0}^{m} A_{n-m-r,\, m-r}\, D_{n-m-r}(z) D_{m-r}(z) \qquad (1)$$

which is valid for arbitrary integral values of m. In the present paper, I propose to look for an allied integral involving the parabolic cylinder function of an arbitrary order, which may represent $D_n(z)$ and may thus furnish a source for an infinite series of the type (1) that may hold for all arbitrary values of m without restriction.

2. *Integral representation of* $D_n(z)$. The general solution of Weber's equation

$$\frac{d^2y}{dz^2} + (n+m+\tfrac{1}{2}-\tfrac{1}{4}z^2)y = 0$$

is given by

$$y = AD_{n+m}(z) + BD_{-n-m-1}(iz)$$

where $D_{n+m}(z)$ and $D_{-n-m-1}(iz)$ are two independent solutions of the same equation. Now taking

$$v = e^{\frac{1}{4}z^2} \int_{\delta} e^{-\frac{1}{4}(z+t)^2} D_n(z+t)\,(-t)^{-m-1} dt$$

we find that v is a solution of the differential equation provided δ the path of integration is so chosen that the quantity $e^{-\frac{1}{4}(z+t)^2} D_n(z+t)\,(-t)^{-m}$ resumes its initial value after describing it. A large number of such paths can be found to fulfil the given condition, each furnishing an integral, which in turn is a solution of Weber's differential equation.

* *Jour. Ind. Math. Soc.*. New Series, Vol. 3, (1939), 226-230.

If $\int_{\infty}^{+0}$ denotes a path which begins at the positive end of the real axis, circulates positively round the origin, and returns back to its starting point, and if $E_{n+m}(z)$ be the integral taken along this path, we have

$$E_{n+m}(z) = e^{\frac{1}{4}z^2}\int_{\infty}^{+0} e^{-\frac{1}{4}(z+t)^2} D_n(z+t)(-t)^{-m-1}dt \qquad (2)$$

as a solution of the differential equation, and consequently a relation of the form

$$E_{n+m}(z) = aD_{n+m}(z) + bD_{-n-m-1}(iz)$$

must exist where a and b are constants to be determined.

Now suppose that arg z has its principal value, so the integrand is rendered a one-valued analytic function by taking $|\arg(-t)| \leqslant \pi$, and consequently $E_{n+m}(z)$ becomes a single-valued analytic function of z, and therefore

$$E_{n+m}(0) = aD_{n+m}(0) + bD_{-n-m-1}(0) \qquad (3.a)$$

and $$E'_{n+m}(0) = aD'_{n+m}(0) + ibD'_{-n-m-1}(0). \qquad (3.b)$$

But $$E_{n+m}(0) = \int_{\infty}^{+0} e^{-\frac{t^2}{4}} D_n(t)(-t)^{-m-1}dt \qquad (4.a)$$

and $$E'_{n+m}(0) = -\int_{\infty}^{+0} e^{-\frac{1}{4}t^2} D_{n+1}(t)(-t)^{-m-1}dt \qquad (4.b)$$

(the differentiation under the sign of integration being permissible).

To evaluate the integrals (4.a) and (4.b), we find that they can be easily deduced by taking $\alpha = \frac{1}{2}$ from Watson's result* *viz.* if $|\arg \alpha| < \frac{\pi}{2}$

then $\int_{\infty}^{(+0)} e^{(\frac{1}{4}-\alpha)z^2} z^m D_n(z)\,dz$

$$= \frac{(\pi)^{\frac{3}{2}} 2^{\frac{n}{2}-m} e^{i(m-\frac{1}{2})\pi}}{\Gamma(-m)\,\Gamma\left(\frac{m}{2}-\frac{n}{2}+1\right)\alpha^{\frac{1}{2}(m+1)}}$$

$$\times {}_2F_1\left(-\frac{n}{2}, \frac{m+1}{2}, \frac{m-n}{2}+1, 1-\frac{1}{2\alpha}\right).$$

Hence

$$E_{n+m}(0) = -\frac{2\pi i}{\Gamma(m+1)} D_{n+m}(0) = aD_{n+m}(0) + bD_{-n-m-1}(0)$$

* *Proc. London, Math. Soc.*, (2) 8, (1910), 393-421.

and $E'_{n+m}(0) = -\frac{2\pi i D'_{n+m}(0)}{\Gamma(m+1)} = aD'_{n+m}(0) + ibD'_{-n-m-1}(0)$

whence we have $a = -\frac{2\pi i}{\Gamma(m+1)}$ and $b = 0$ and consequently the relation

$$D_{n+m}(z) = -\frac{e^{\frac{1}{4}z^2}\Gamma(m+1)}{2\pi i}\int_{\infty}^{+0} e^{-\frac{1}{4}(z+t)^2} D_n(z+t)(-t)^{-m-1} dt \tag{5}$$

which, on deforming the path of integration, when $R(m) < 0$, can be expressed as

$$\Gamma(-m) D_{n+m}(z) = e^{\frac{1}{4}z^2}\int_0^{\infty} e^{-\frac{1}{4}(z+t)^2} D_n(z+t) t^{-m-1} dt. \tag{6}$$

On interchanging n and m, we obtain from (5)

$$D_{n+m}(z) = -\frac{e^{-\frac{1}{4}z^2}\Gamma(n+1)}{2\pi i}\int_{\infty}^{+0} e^{-\frac{1}{4}(z+t)^2} D_m(z+t)(-t)^{-n-1} dt \tag{7}$$

which for $n = n - m$, assumes the form

$$D_n(z) = -\frac{\Gamma(n-m+1)}{2\pi i} e^{\frac{1}{4}z^2}\int_{\infty}^{+0} e^{-\frac{1}{4}(z+t)^2} D_m(z+t)(-t)^{m-n-1} dt, \tag{8}$$

and expresses $D_n(z)$ by means of an integral involving a parabolic cylinder function of an arbitrary order m.

As a particular case of this general result for $m = 0$, we have

$$D_n(z) = -\frac{\Gamma(n+1)}{2\pi i} e^{-\frac{1}{4}z^2}\int_{\infty}^{+0} e^{-\frac{t^2}{2} - zt}(-t)^{-n-1} dt, \tag{9}$$

a well known integral* due to E. T. Whittaker.

3. *Expansion of $D_n(z)$ into an infinite series.*

Since $\frac{d^m}{dz^m}\left[e^{\frac{1}{4}z^2} D_n(z)\right] = n(n-1)\dots(n-m+1) e^{\frac{1}{4}z^2} D_{n-m}(z)$

we have by Taylor's Theorem

$$e^{\frac{1}{4}(u+v)^2} D_n(u+v) = e^{\frac{1}{4}u^2}\Gamma(n+1)\sum_{r=0}^{\infty}\frac{D_{n-r}(u) v^r}{\Gamma(n-r+1) r!} \tag{10}.$$

* E. T. Whittaker, *Proc. Lond. Math. Soc.*, (1) 35 (1903), 417-427.

Hence from (5), we obtain

$$D_{n+m}(z) = -\frac{\Gamma(m+1)}{2\pi i} e^{\frac{1}{4}z^2}$$

$$\times \int_{\infty}^{+0} e^{-\frac{1}{2}(z+t)^2 + \frac{z^2}{4}} \Gamma(n+1) \sum_{r=0}^{\infty} \frac{D_{n-r}(z)(-t)^{-m-1+r}(-1)^r dt}{\Gamma(n-r+1)r!}$$

$$= -\frac{\Gamma(m+1)\ \Gamma(n+1)}{2\pi i} \left[\sum_{r=0}^{\infty} \frac{(-1)^r D_{n-r}(z)}{\Gamma(n-r+1)r!} \times \int_{\infty}^{+0} e^{-\frac{t^2}{2} - zt} (-t)^{-m-1+r} dt \right]$$

which after integrating term by term, which is possible, yields by (9) the result,

$$e^{-\frac{1}{4}z^2} D_{n+m}(z) = \Gamma(m+1)\ \Gamma(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r D_{n-r}(z) D_{m-r}(z)}{\Gamma(n-r+1)\ \Gamma(m-r+1) r!} \tag{11}$$

i.e. $$e^{-\frac{1}{4}z^2} D_{n+m}(z) = \sum_{r=0}^{\infty} A_{n-r,\, m-r}\, D_{n-r}(z) D_{m-r}(z)$$

where $$A_{n-r,\, m-r} = \frac{(-1)^r \Gamma(n+1)\ \Gamma(m+1)}{\Gamma(n-r+1)\ \Gamma(m-r+1)}$$

$$= \frac{(-1)^r n(n-1)\ldots(n-r+1)m(m-1)\ldots(m-r+1)}{r!}.$$

In case m is negative, say $-m$, starting from (6), we similarly obtain

$$e^{-\frac{1}{4}z^2} D_{n-m}(z) = \sum_{r=0}^{\infty} \frac{n(n-1)\ldots(n-r+1)m(m+1)\ldots(m+r-1)}{r!} \times D_{n-r}(z) D_{-m-r}(z)$$

which is the same as

$$e^{-\frac{1}{4}z^2} D_{n-m}(z) = \sum_{r=0}^{\infty} A_{n-r,\, -m-r} D_{n-r}(z) D_{-m-r}(z), \tag{12}$$

a particular case of (11) when m is replaced by $-m$.

* The result (10) was obtained by Gorakh Prasad by a different method. *Proc. B. M. S.*. 4 (1922).

Hence we conclude that for all values of m and n

$$e^{-\frac{1}{4}z^2} D_{n+m}(z) = \sum_{r=0}^{\infty} A_{n-r,\, m-r} D_{n-r}(z) D_{m-r}(z).$$

In case n or m is integral the series terminates. Changing n to $n-m$, the result can be expressed in the form

$$e^{-\frac{1}{4}z^2} D_n(z) = \sum_{r=0}^{\infty} A_{n-m-r,\, m-r} D_{n-m-r}(z) D_{m-r}(z), \quad (13)$$

a relation that is true for all values of m. For integral values of m, (13) degenerates into (1).

ON A PROBLEM OF ARRANGEMENTS (I)

By V. NARASIMHA MURTI, *Andhra University.*

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Introduction:—The following problem was proposed by Dr. Vijayaraghavan to Gul Abdulla and Lal Bahadur* who have established the possibility of its solution when $2n+1$ is a prime number, and have also given the solution of the problem when $2n+1=21$, a composite number. Later Dr. Hansaraj† Gupta has shown how from a solution for $n=m-1$, a solution for $n=m$ can in general be constructed. In proving this, he assumed an unproved problem in permutations, whose solution 'seems always to exist'. By using Gupta's method, Dr. S. Chowla‡ has proved that the problem is solvable for $n=m+1$ where $2n-1$ is a prime number. But Prof. F. W. Levi has obtained a simple proof of the general problem which is not yet published. The following is an independent proof for all $2n+1$, composite or prime.

1. The Problem. $2n+1$ *persons are invited to a dinner on n different days. Is it possible to arrange them on a circular table in such a way that no person has the same neighbour on different days?*

Let us call the problem briefly as P_n. Let us represent the $2n+1$ persons by the numbers $1, 2, \ldots, 2n+1$; and let $(a_1, a_2, \ldots, a_{2n+1})$, where $a_1, a_2, \ldots, a_{2n+1}$ are the numbers $1, 2, \ldots, 2n+1$ in some order, denote an arrangement, in which a_{s-1} and a_{s+1} are neighbours to a_s for $2 \leqslant s \leqslant 2n$, a_{2n} and a_1 are neighbours to a_{2n+1}, and a_2 and a_{2n+1} are neighbours to a_1.

Suppose the following is a solution (let us call it S_n) of P_n:—

$$(y_{n+1}+n, y_n+n, \ldots, y_2+n, y_1, y_2, y_3, \ldots, y_{n+1})\text{—1st day},$$

$$(y_{n+1}+n+1, y_n+n+1, \ldots, y_2+n+1, y_1, y_2+1, y_3+1, \ldots, y_{n+1}+1)\text{—2nd day},$$

$$(y_{n+1}+n+n-1, y_n+n+n-1, \ldots, y_2+n+n-1, y_1, y_2+n-2, y_3+n-1, \ldots, y_{n+1}+n-1)\text{—}n\text{th day}.$$

**Proc. Ind. Acad. Sc.* 9 (1939), 103.
†*Proc. Ind. Acad. Sc.* 9 (1939), 399-403.
‡*Proc. Ind. Acad. Sc.* 9 (1939), 418-21.

Here, we consider any number congruent to s modulo $2n$ as s itself when $2 \leqslant s \leqslant 2n-1$; any number congruent to 1 modulo $2n$ as $2n+1$; and any number congruent to 0 modulo $2n$ as $2n$ itself. In that way we cover up all the $2n$ numbers from 2 up to $2n+1$.

In the above solution y_1 and y_2 are fixed and their values are taken to be 1 and 2 respectively, so that the neighbours to 1 on the 1st day are 2 and $2+n$, on the second day 3 and $3+n$, and on the nth day $n+1$ and $2n+1$. Thus the $2n$ remaining persons are distributed in pairs as neighbours to the person 1, during all the n days.

Now let us put $y_1 = 1$, $y_2 = 2$, $y_3 = 2+x_1, \ldots,$

$y_n = 2+x_1+x_2+\ldots+x_{n-2}$, $y_{n+1} = 2+x_1+x_2+\ldots+x_{n-1}$.

Without loss of generality, all the x's are less than $2n$.

2. THEOREM I. *The necessary and sufficient condition for any one arrangement of s_n in* **1** *to contain all the numbers* $1, 2, \ldots, 2n+1$ *in some order is that*

$$x_m+x_{m+1}+\ldots+x_{m+t} \not\equiv 0 \text{ (modulo } n) \qquad (1)$$

$m = 1, 2, \ldots;\ t = 0, 1, \ldots;\ 1 \leqslant m+t \leqslant u-1.$

PROOF:—(i) Suppose that the condition (1) does not hold. Let $x_m+x_{m+1}+\ldots+x_{m+t}$ be an odd multiple of n. Then the two numbers $y_2+x_2+\ldots+x_{m-1}+n$, and $y_2+x_1+\ldots+x_{m+t}$, which are the numbers belonging to the first row of s_n, will be equal or congruent modulo $2n$. Therefore $x_m+x_{m+1}+\ldots+x_{m+t}$ cannot be an odd multiple of n.

Next suppose $x_m+x_{m+1}+\ldots+x_{m+t}$ is an even multiple of n Then the two numbers $y_2+x_1+\ldots+x_{m-1}$ and $y_2+x_1+\ldots+x_{m-1}+x_m+\ldots+x_{m+t}$, of the first row, will either be equal or congruent modulo $2n$. Hence it follows that $x_m+\ldots+x_{m+1}$ cannot be an even multiple of n. Therefore the condition (1) of the theorem is necessary.

(ii) Conversely, suppose the condition (1) is satisfied. Then no two numbers lying to the right of the number y_1, or 1, can be congruent modulo $2n$, on account of the restriction $x_m+x_{m+1}+\ldots+x_{m+t} \neq$ an even multiple of n. Consequently no two members lying to the left of y_1 can be congruent modulo $2n$. Secondly, since by the condition (1), $x_m+x_{m+1}+\ldots+x_{m+t} \neq$ an odd multiple of n, the possibility of any number like $y_1+x_1+x_2+\ldots+x_{m+t}$ (lying on the right side of y_1) becoming congruent to $y_1+x_1+\ldots+x_{m-1}+n$ modulo $2n$ (lying on the left side of y_1) is excluded.

And since there is no other possibility for the numbers to become equal or congruent modulo $2n$, the condition (1) is sufficient to show that all the $2n+1$ numbers of the first row are different.

The same reasoning can be applied even when we take any other row of s_n.

THEOREM II. *The necessary and sufficient conditions that no two neighbours appearing in one arrangement appear again as neighbours in some other arrangement of the solution S_n are* $x_r \neq x_s$, $r \neq s$, (2)

and $x_r + x_s \not\equiv 0 \pmod{2n}$, $1 \leqslant r, s \leqslant n-1$. (3)

PROOF. (i) Suppose the condition (2) is not satisfied. Then $x_r = x_s$ $(r \neq s)$. Let $r < s$. It is evident that for a value k_1 of k, $y_2+x_1+x_2+\ldots+x_{r-1}+k$ is equal to $y_2+x_1+x_2+\ldots+x_{s-1}+l$ for some value l_1 of l, k and l taking the values $0, 1, \ldots, 2n-1$. Then the two pairs of numbers $(y_2+x_1+x_2+\ldots+x_{r-1}+k_1,\ y_2+x_1+\ldots+x_{r-1}+x_r+t_1)$, and $(y_2+x_1+\ldots+x_{s-1}+l_1,\ y_2+x_1+\ldots+x_{s-1}+x_s+l_1)$ will be the same, since $x_r = x_s$.

Next suppose the condition (3) is not true. Then $x_r+x_s \equiv 0$ (modulo $2n$). Since one of the numbers $y_2+x_1+\ldots+x_{r-1}+k$ $(k = 0, 1, \ldots, 2n-1)$ is equal to one of the numbers $y_2+x_1+\ldots+x_{s-1}+x_s+l$, $(l = 0, 1, 2, \ldots, 2n-1)$, for some k_1 and l_1, the following two pairs of numbers

$$(y_2+x_1+\ldots+x_{r-1}+k_1,\ y_2+x_1+\ldots+x_{r-1}+x_r+k_1)$$

and $$(y_2+x_1+\ldots+x_{s-1}+x_s+l_1,\ y_2+x_1+\ldots+x_{s-1}+l_1)$$

will be the same, since $x_r \equiv -x_s$ (modulo $2n$).

Hence the conditions (2) and (3) of this theorem are necessary.

(ii) Suppose the conditions (2) and (3) are satisfied. Then consider the two pairs of numbers

$$(y_2+x_1+\ldots+x_{r-1}+k,\ y_2+x_1+\ldots+x_{r-1}+x_r+k), \quad (k = 0, 1, \ldots, 2n-1).$$

and

$$(y_2+x_1+\ldots+x_{s-1}+l,\ y_2+x_1+\ldots+x_{s-1}+x_s+l), \quad (l = 0, 1, \ldots, 2n-1).$$

For some $k = k_1$ and $l = l_1$, let these two pairs be the same. Now, if

$$y_2+x_1+\ldots+x_{r-1}+k_1 \equiv y_2+x_1+\ldots+x_{s-1}+l_1 \quad (\text{modulo } 2n)$$

then, $y_2+x_1+\ldots+x_{r-1}+x_r+k_1$
must also be congruent to

$$y_2+x_1+\ldots+x_{s-1}+x_s+l_1 \pmod{2n}.$$

From these two congruences it follows that $x_r \equiv x_s$ (modulo $2n$), which is contrary to the condition (2).

Again, if

$$y_2+x_1+\ldots+x_{r-1}+k_1 \equiv y_2+x_1+\ldots+x_{s-1}+x_s+l_1 \pmod{2n}$$

then, $y_2+x_1+\ldots+x_{r-1}+x_r+k_1 \equiv y_2+x_1+\ldots+x_{s-1}+l_1$ (modulo $2n$).

From these two congruences, again, it follows that $x_r \equiv -x_s$ (modulo $2n$), which is contrary to the condition (3).

Hence we have proved the sufficiency of the conditions (2) and (3).

To sum up, the condition (1) of Theorem I ensures that all the $2n+1$ numbers in each arrangement of s_n are distinct, and the conditions (2) and (3) of Theorem II exclude the possibility of any two neighbours in one arrangement appearing again as neighbours in another row. So when these three conditions are satisfied between the x's, we have established the existence of the solution S_n of the problem P_n.

3. *Illustrations*:—Let us set $x_1=1$, $x_2=2n-2$, $x_3=3$, $x_4=n-4$, $x_5=5, \ldots, x_{n-1}=n-1$ or $n+1$ according as n is even or odd. Then

$$x_m+x_{m+1}+\ldots+x_{m+t}$$

$$\equiv \pm\left\{m-(m+1)+(m+2)-\ldots+(-1)^t(m+t)\right\} \pmod{n},$$

according as m is odd or even

$$\equiv \pm\left\{-\frac{t+1}{2}\right\} \pmod{n} \quad \textit{when } t \textit{ is odd}$$

which is $\not\equiv 0 \pmod{n}$ since $1 \leqslant m+t \leqslant n-1$,

and $$\equiv \pm\left\{-\frac{t}{2}+(m+t)\right\} \pmod{n} \quad \textit{when } t \textit{ is even}$$

which is $\not\equiv 0 \pmod{n}$ since $1 \leqslant m+t \leqslant n-1$.

Therefore the condition (1) of Theorem I is satisfied. It is obvious that the conditions (2) and (3) of Theorem II are also satisfied between these x's.

By reversing the order of the x's we can obtain another solution in which $x_{n-1} = 1$, $x_{n-2} = 2n-2, \ldots, x_1 = n-1$ or $n+1$ according as n is even or odd.

When n is a power of 2, the set of x's given by $x_1 = 1$, $x_2 = 2, \ldots, x_{n-1} = n-1$ satisfy the conditions of the above Theorems. For,

$$x_m + x_{m+1} + \ldots + x_{m+t} = m + (m+1) + \ldots + (m+t)$$

$$= \frac{(2m+t)(t+1)}{2}, \quad (1 \leqslant m+t \leqslant n-1)$$

is not a multiple of n, since *for t odd*, $2m+t$ is odd and $(t+1)/2$ is an integer $< n$, and *for t even*, $t+1$ is odd and $(2m+t)/2$ is an integer $< n$.

The sum of any two x's is less than $2n$, an[illegible] x's are equal.

Examples. $n = 10$. $x_1 = 1$, $x_2 = 18$, $x_3 = 3$, $x_4 =$ 1[illegible], $x_6 = 14$, $x_7 = 7$, $x_8 = 12$, $x_9 = 9$.

Then the first row of S_{10} will be (17, 8, 16, 9, 15, 10, 14, 11, 13, 12, 1, 2, 3, 21, 4, 20, 5, 19, 6, 18, 7).

Also, $x_1 = 9$, $x_2 = 12$, $x_3 = 7$, $x_4 = 14$, $x_5 = 5$, $x_6 = 16$, $x_7 = 3$, $x_8 = 18$, $x_9 = 1$. Then the first row of S_{10} will be (17, 16, 18, 15, 19, 14, 20, 13, 21, 12, 1, 2, 11, 3, 10, 4, 9, 5, 8, 6, 7,).

$n = 8 = 2^3$. $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $x_5 = 5$, $x_6 = 6$, $x_7 = 7$.

Then the first row of S_8 will be (6, 15, 9, 4, 16, 13, 11, 10, 1, 2, 3, 5, 8, 12, 17, 7, 14).

Reversing the order of the x's we get another S_8.

I express my thanks to Dr. V. Ramaswami for his helpful guidance.

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